

RESOLVING SHARE AND TOPOLOGICAL INDEX

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ABSTRACT. An atom a of a molecular graph G uniquely determines (resolves) a pair (a_1, a_2) of atoms of G if the distance between a and a_1 is different from the distance between a and a_2 . In this paper, we quantify the involvement of each atom a of G in uniquely determining (resolving) a pair (a_1, a_2) of atoms of G , which is called the resolving share of a for the pair (a_1, a_2) . Using this quantity, we define a distance-based topological index of a molecular graph, which reflects the topology of that molecular graph according to the resolvability behavior of each of its atom, and is called the resolving topological index. Then we compute the resolving topological index of several molecular graphs.

1. INTRODUCTION

A major part of the current research in mathematical chemistry, chemical graph theory and quantitative structure-activity-property relationship QSAR/QSPR studies involves topological indices. Topological indices are numerical identities derived in an unambiguous manner from a molecular graph [22, 29]. These indices are graph invariants which usually characterize the topology of that molecular graph. Some major classes of topological indices such as distance-based topological indices, connectivity topological indices and counting related polynomials and indices of graphs have found remarkable employment in several chemistry fields.

The first non-trivial distance-based topological index was Wiener index, introduced by Wiener in 1947 [30]. To explain various chemical and physical properties of atoms, molecules, and to correlate the structure of molecules to their biological activity, Wiener index plays a significant role [19]. Caused by this usefulness of the Wiener index, the research interest in Wiener index and related distance-based indices is still considerable. In the last twenty years, surprisingly a large number of modifications and extensions of the Wiener index such as Schultz index $MTI(G)$, proposed by Schultz [27]; Szeged index $S_z(G)$, proposed by Gutman [7]; revised Wiener or revised Szeged index $S_{z^*}(G)$ proposed by Randić [21]; modified Wiener

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index for trees ${}^mW(T)$, proposed by Nikolić *et al.* [18]; another class of modified Wiener indices ${}^mW_\lambda(T)$, proposed by Gutman *et al.* [8]; Harary index $H(G)$, proposed by Plavšić *et al.* [20] and Baladan index $J(G)$, proposed by Baladan [2] and by Randić [23], to name a few, was put forward and studied. An extensive bibliography on this matter can be found in the reviews [6, 15].

The problems on distance in graphs continues to seek the attention of scientists both as theory and applications. Among these problems, the most famous problem in graphs, which plays a vital role to uniquely distinguish all the vertices of a graph, is resolvability. Roughly speaking, by resolvability in a graph G , we mean that any two vertices of G in the pair (u, v) are said to be uniquely distinguished (represented or resolved) by a vertex w of G if the distance between u and w is distinct from the distance between v and w . The vertices of G needed to distinguish all the pairs of the vertices of G form a remarkable set known as a resolving set for G , and it has a significant application in pharmaceutical chemistry.

A fundamental problem in pharmaceutical chemistry is to find out the unique representations of chemical compounds in several molecular structures (graphs). The intention behind uniquely representing the chemical compounds is to determine whether any two compounds in the structure share the same functional group at a particular position. This comparative statement plays a fundamental part in drug discovery whenever it is to be determined whether the features of a chemical compound are responsible for its pharmacological activity [12, 13]. The solution of this fundamental problem was addressed by the concept of resolving set. A minimum resolving set is, in fact, the set of those few atoms in a molecular graph which determine the unique representations of the chemical compounds. Now, a question with remarkable interest arises that *how much an atom of a molecular graph partake in uniquely representing any pair of chemical compounds?* Precisely, in a graph G , how much a vertex of G involves itself to resolve any pair of vertices of G ? In this paper, we fix this problem by defining the amount of resolving done by an atom (vertex) to represent (resolve) every pair of compounds of a molecular graph, and is called the resolving share of that atom. With the help of resolving share of each atom, a numeric identity is associated with the molecular graph, called the resolving topological index, which reflects the total amount of resolving done by the atoms in that molecular graph.

Under a “molecular graph” we understand a simple graph, representing the atom skeleton of molecules (chemical compounds). Thus the vertices of a molecular graph represents the atoms and edges the atom-atom bonds. Let G be a non-trivial connected graph with vertex set $V(G)$ and edge set $E(G)$. We write $u \sim v$ if two vertices u and v are adjacent (form an edge) in G and write $u \not\sim v$ if they are non-adjacent (do not form an edge). The join of two graphs G_1 and G_2 , denoted by $G_1 + G_2$, is a graph with vertex set $V(G_1) \cup V(G_2)$ and an edge set

$E(G_1) \cup E(G_2) \cup \{u \sim v \mid u \in V(G_1) \wedge v \in V(G_2)\}$. The *distance*, $d(u, v)$, between two vertices u and v of G is defined as the length of a shortest $u - v$ path in G , where length is the number of edges in the path. The *diameter* of G , denoted by $diam(G)$, is the maximum distance between any two vertices of G . We refer [4] for the general graph theoretic notations and terminologies not described in this paper.

A vertex u of G *resolves* two distinct vertices v and w of G if $d(v, u) \neq d(w, u)$. A set $R \subseteq V(G)$ is called a *resolving set* for G if every two distinct vertices of G are resolved by some elements of R . Such a set R with minimum cardinality is called a *metric basis*, or simply a *basis* of G and that minimum cardinality is called the *metric dimension* of G , denoted by $\dim(G)$ [3]. Obviously, the metric dimension of a graph G is a topological index that suggests those minimum number of vertices of G which uniquely determine all the vertices of G by their shortest distances to the chosen vertices.

The concept of resolving set was first introduced in the 1970s, by Slater [28] and, independently, by Harary and Melter [9]. Slater described the usefulness of this idea into long range aids to navigation [28]. Moreover, this concept has some applications in chemistry for representing chemical compounds [12, 13] and in problems of pattern recognition and image processing, some of which involve the use of hierarchical data structures [16]. Other applications of this concept to navigation of robots in networks and other areas appear in [3, 14]. In recent years, a considerable literature regarding this notion has developed (see [1, 3, 5, 10, 11, 14, 16, 17, 24, 25, 26]).

2. RESOLVING SHARE

In this section, we define the concept of resolving share and investigate some basic results which later help in defining and computing a distance-based topological index. We begin with the following useful preliminaries: V_p denotes the collection of all $\binom{n}{2}$ pairs of the vertices of a graph G . For any vertex w of G , let $V_i(w) = \{v \in V(G) - \{w\} \mid d(v, w) = i\}$ be the distance neighborhood of w for $1 \leq i \leq diam(G)$, and the partition

$$\Pi_w = \{V_i(w) ; 1 \leq i \leq diam(G)\}$$

be the distance partition of the set $V(G) - \{w\}$ with reference of w . By $\Pi_w - \{x\}$, we mean that the vertex x is not lying in any partite set of the distance partition Π_w . By $\Pi_x = \Pi_y$, we mean that $V_i(x) = V_i(y)$ for all $1 \leq i \leq diam(G)$.

Definition 2.1. Let G be a connected graph. For any pair $(u, v) \in V_p$, let $R(u, v) = \{x \in V(G) \mid x \text{ resolves } u \text{ and } v\}$ be the resolving neighborhood of the pair (u, v) . Then for any $w \in V(G)$, the quantity

$$r_w(u, v) = \begin{cases} \frac{1}{|R(u, v)|} & \text{if } u \text{ and } v \text{ are resolved by } w, \\ 0 & \text{otherwise} \end{cases}$$

is called the resolving share of w for the pair (u, v) .

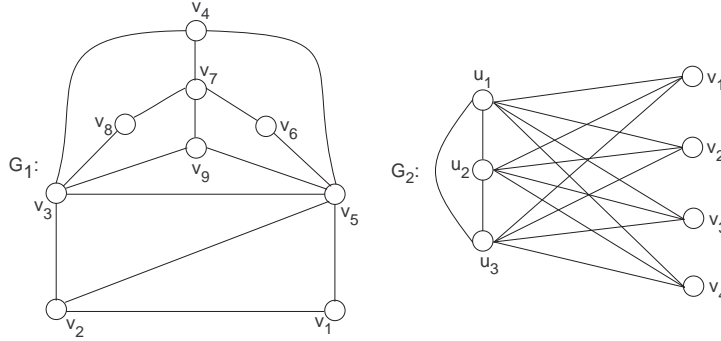


FIGURE 1. In the graph G_1 , the resolving share of the vertex v_1 is zero for the pair (v_2, v_5) ; is $\frac{1}{4}$ for the pair (v_2, v_4) ; is $\frac{1}{5}$ for the pair (v_2, v_6) and is $\frac{1}{9}$ for the pair (v_2, v_7) .

Remarks 2.2. (i) The resolving neighborhood of a pair $(u, v) \in V_p$ is the class of all those vertices whose resolving share for the pair (u, v) is same.

(ii) For $w \in V(G)$ and $(u, v) \in V_p$,

$$0 \leq r_w(u, v) \leq \frac{1}{2}.$$

(iii) $r_u(u, v) \neq 0 \neq r_v(u, v)$.

Lemma 2.3. For a pair $(u, v) \in V_p$ and for a vertex $w \in V(G) - \{u, v\}$, $r_w(u, v) = 0$ if and only if both u and v belong to the same partite set of Π_w .

Proof. (\Rightarrow) $r_w(u, v) = 0$ implies that $d(u, w) = d(v, w) = i$ for some $1 \leq i \leq \text{diam}(G)$. It follows that both u and v belong to the same partite set $V_i(w) \in \Pi_w$.

(\Leftarrow) If $u, v \in V_i(w) \in \Pi_w$ for some $1 \leq i \leq \text{diam}(G)$, then w does not resolve u and v , and hence $r_w(u, v) = 0$. \square

Lemma 2.4. For a pair $(u, v) \in V_p$ and for all $w \in V(G) - \{u, v\}$, $r_w(u, v) = 0$ if and only if $\Pi_u - \{v\} = \Pi_v - \{u\}$.

Proof. (\Rightarrow) Suppose that $r_w(u, v) = 0$ for all $w \in V(G) - \{u, v\}$. This implies that $d(u, w) = d(v, w)$ for all $w \in V(G) - \{u, v\}$. Contrarily assume that $\Pi_u - \{v\} \neq \Pi_v - \{u\}$. It follows that there exists an element x in $V(G) - \{u, v\}$ such that x lies in a partite set, say $V_i(u)$, of $\Pi_u - \{v\}$ and x lies in a partite set, say $V_j(v)$ ($j \neq i$), of $\Pi_v - \{u\}$, and vice-versa. Thus $d(x, u) = i \neq d(x, v)$ or $d(x, v) = j \neq d(x, u)$, a contradiction. Hence $\Pi_u - \{v\} = \Pi_v - \{u\}$.

(\Leftarrow) Suppose that $\Pi_u - \{v\} = \Pi_v - \{u\}$. Assume contrarily that $r_w(u, v) \neq 0$. It follows that $w \in R(u, v)$ and hence $d(u, w) \neq d(v, w)$. Thus, there exists a partite set in $\Pi_u - \{v\}$ which is not equal to any members of $\Pi_v - \{u\}$, a contradiction. Hence $r_w(u, v) = 0$. \square

Lemma 2.5. *For a pair $(u, v) \in V_p$ and for a vertex $w \in V(G)$, $r_w(u, v) = \frac{1}{2}$ if and only if $w \in \{u, v\}$ and $\Pi_u - \{v\} = \Pi_v - \{u\}$.*

Proof. (\Rightarrow) Suppose that $r_w(u, v) = \frac{1}{2}$. Then, clearly, $w \in \{u, v\}$ because $|R(u, v)| = 2$ and w resolves u and v . In fact, $R(u, v) = \{u, v\}$ in this case. It follows that $d(u, x) = d(v, x)$ for all $x \in V(G) - \{u, v\}$, which concludes that $\Pi_u - \{v\} = \Pi_v - \{u\}$.

(\Leftarrow) Suppose that $\Pi_u - \{v\} = \Pi_v - \{u\}$. It follows that $d(u, z) = d(v, z)$ for all $z \in V(G) - \{u, v\}$, which implies that the only vertices that resolves the pair (u, v) are the vertices in the pair. So $R(u, v) = \{u, v\}$, and hence for $w \in \{u, v\}$, $r_w(u, v) = \frac{1}{2}$. \square

Example 2.6. *Consider the graph G_2 of Figure 1 with vertex set $V(G_2) = U = \{u_1, u_2, u_3\} \cup V = \{v_1, v_2, v_3, v_4\}$. Let $(u, v) \in V_p$ be any pair of vertices of G_2 . Then note that, (i) $\Pi_u - \{v\} = \Pi_v - \{u\}$ for either $u, v \in U$ or $u, v \in V$; (ii) if $u \in U$ and $v \in V$, then $R(u, v) = \{u\} \cup V$ and both u, v belong to the same partite set of Π_x for all $x \in U - \{u\}$. Hence, by previous three lemmas, we have*

$$r_w(u, v) = \begin{cases} \frac{1}{2} & \text{if } w \in \{u, v\}, \text{ for either } u, v \in U \text{ or } u, v \in V, \\ \frac{1}{5} & \text{if } w \in \{u\} \cup V, \text{ for } u \in U \text{ and } v \in V, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 2.7. *For each pair $(u, v) \in V_p$, $R(u, v) \cap R \neq \emptyset$ for any resolving set R for a graph G .*

The following useful result for finding a resolving set for G was proposed by Chartrand *et al.* in 2000.

Lemma 2.8. [3] *Let R be a resolving set for a graph G and $(u, v) \in V_p$. If $d(u, w) = d(v, w)$ for all $w \in V(G) - \{u, v\}$, then u or v is in R .*

Lemma 2.9. *Let a pair $(u, v) \in V_p$ and R be any resolving set for G .*

- (1) *If $r_w(u, v) = 0$ for all $w \in V(G) - \{u, v\}$, then u or v is in R .*
- (2) *If $r_w(u, v) = \frac{1}{2}$, then u or v is in R .*

Proof. (1) By Lemma 2.4, $\Pi_u - \{v\} = \Pi_v - \{u\}$, which implies that $d(u, w) = d(v, w)$ for all $w \in V(G) - \{u, v\}$, and hence Lemma 2.8 yields the result.

(2) By Lemma 2.5, $\Pi_u - \{v\} = \Pi_v - \{u\}$ and $w \in \{u, v\}$. In fact $R(u, v) = \{u, v\}$ and $d(u, w) = d(v, w)$ for all $w \in V(G) - \{u, v\}$. Hence, the result follows by Lemma 2.8. \square

Let $|G|$ denotes the order of a graph G . The following assertion is directly follows from the definition of the resolving share.

Proposition 2.10. *For a pair $(u, v) \in V_p$ and for each vertex $w \in V(G)$, $r_w(u, v) = \frac{1}{|G|}$ if and only if $R(u, v) = V(G)$.*

Lemma 2.11. *For a pair $(u, v) \in V_p$, if $r_w(u, v) = \frac{1}{|G|}$ for each $w \in V(G)$, then the distance between u and v is odd.*

Proof. If the distance between u and v is even, i.e., $d(u, v) = 2k$ for $k \geq 1$, then there exists a vertex x in G such that $d(u, x) = k = d(x, v)$, and hence $x \notin R(u, v)$. \square

Theorem 2.12. *Let G be a graph with $\text{diam}(G) = 2$. Then there are at most $\left\lfloor \left(\frac{|G|}{2}\right)^2 \right\rfloor$ pairs (u, v) in V_p for which $r_w(u, v) = \frac{1}{|G|}$ for each $w \in V(G)$.*

Proof. Let $(u, v) \in V_p$ for which $r_w(u, v) = \frac{1}{|G|}$ for each $w \in V(G)$. Then, $R(u, v) = V(G)$, by Proposition 2.10, and since $\text{diam}(G) = 2$ so $d(u, v) = 1$, by Lemma 2.11. It follows that $u \in V_1(v) \in \Pi_v$ and $v \in V_1(u) \in \Pi_u$. Moreover, $V_1(u)$ and $V_1(v)$ are disjoint subsets of $V(G)$, because if there is an element x in $V_1(u) \cap V_1(v)$, then x does not resolve u and v and hence $R(u, v) \neq V(G)$. Also, $V_1(u) \cup V_1(v) = V(G)$. Otherwise, there exists an element y in $V(G) - \{u, v\}$ such that $u, v \in V_2(y)$, which implies that u and v are not resolved by y yielding $R(u, v) \neq V(G)$. Hence $V_1(u)$ and $V_1(v)$ form a partition of $V(G)$. Further, for any $a, b \in V_1(u)$ (or $a, b \in V_1(v)$), a and b have the same distance from u (or v) and hence $R(a, b) \neq V(G)$. It follows that the number of pairs (u, v) for which $R(u, v) = V(G)$ is bounded above by

$$|V_1(u)||V_1(v)| \leq \left\lfloor \left(\frac{|G|}{2}\right)^2 \right\rfloor.$$

\square

Lemma 2.13. *If the diameter of a graph G is one, then for every pair $(u, v) \in V_p$, $r_w(u, v) \in \{0, \frac{1}{2}\}$ for each $w \in V(G)$.*

Proof. Since $\text{diam}(G) = 1$, so G is isomorphic to a complete graph, and for every two vertices u and v of a complete graph, $\Pi_u - \{v\} = \Pi_v - \{u\}$. Hence, the result followed by Lemma 2.4 if $w \notin \{u, v\}$, or followed by Lemma 2.5 if $w \in \{u, v\}$. \square

Theorem 2.14. *Let G be a graph. Then for every pair $(u, v) \in V_p$ and for each $w \in V(G)$,*

$$r_w(u, v) = \begin{cases} \frac{1}{2} & \text{if } w \in \{u, v\}, \\ 0 & \text{if } w \notin \{u, v\}, \end{cases}$$

if and only if $\text{diam}(G) = 1$.

Proof. Suppose that

$$r_w(u, v) = \begin{cases} \frac{1}{2} & \text{if } w \in \{u, v\}, \\ 0 & \text{if } w \notin \{u, v\}, \end{cases}$$

for every pair $(u, v) \in V_p$ and for each $w \in V(G)$. It follows that $R(u, v) = \{u, v\}$ and $w \in R(u, v)$ or $w \notin R(u, v)$. We claim that $\text{diam}(G) = 1$. Suppose that

$d(u, v) = k \geq 2$ and $u, u_1, u_2, \dots, u_{k-1}, v$ be a $u - v$ geodesic (shortest path) in G of length k . This implies that $d(u_1, u) = 1$ and $d(u_1, v) = k - 1$, and hence $u_1 \in R(u, v)$. But $u_1 \notin \{u, v\}$, a contradiction to the fact that $R(u, v) = \{u, v\}$. Consequently $d(u, v) = 1$ for every two distinct vertices u, v of G . So $\text{diam}(G) = 1$.

The converse part of the theorem followed by Lemma 2.13. \square

3. RESOLVING TOPOLOGICAL INDEX

We define the average resolve share of each vertex of a graph and then by using it we establish a distance based topological index of that graph. Further, we compute the resolving topological index of certain graphs.

Definition 3.1. Let G be a connected graph. For any vertex $w \in V(G)$, let $R(w) = \{(u, v) \in V_p \mid u \text{ and } v \text{ are resolved by } w\}$ be the resolvent neighborhood of the vertex w . Then the quantity

$$ar_w(G) = \frac{1}{|R(w)|} \sum_{(u,v) \in R(w)} r_w(u, v)$$

is the average of the amount of resolving done by w in G , and is called the average resolving share of w in G .

Remark 3.2. Since for each $x \in V(G) - \{w\}$, $(w, x) \in R(w)$, so $|R(w)|$ and $ar_w(G)$ will never zero for every $w \in V(G)$.

Since $\sum_{(u,v) \in R(w)} r_w(u, v) = |R(w)|a \Leftrightarrow r_w(u, v) = a$ for all $(u, v) \in R(w)$. So, we have the following straightforward proposition:

Proposition 3.3. Let G be a graph. For any $w \in V(G)$, $ar_w(G) = a$ if and only if $r_w(u, v) = a$ for all $(u, v) \in R(w)$.

Proposition 3.4. Let G be a graph and w be any vertex of G . If each partite set of the distance partition Π_w is a singleton set, then

$$ar_w(G) = \frac{2}{|G|(|G| - 1)} \sum_{(u,v) \in V_p} r_w(u, v).$$

Proof. If each partite set of the distance partition Π_w is a singleton set, then the vertices of each pair $(u, v) \in V_p$ are resolved by w . It follows that $R(w) = V_p$ and the proof is complete. \square

Definition 3.5. Let G be a graph. Then the positive real number

$$\mathcal{R}(G) = \sum_{w \in V(G)} ar_w(G) \quad \text{is called the resolving topological index of } G.$$

Theorem 3.6. The resolving topological index of the Petersen graph is $\frac{5}{3}$.

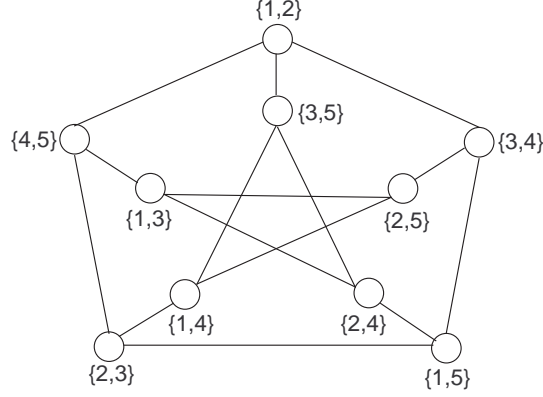


FIGURE 2. The Petersen graph

Proof. Let G be the Petersen graph. The vertices of G are the 2-element subsets of the set $\{1, 2, 3, 4, 5\}$. Let the vertex set of G be $\{S_{ij} = \{i, j\} ; 1 \leq i < j \leq 5\}$ and two subsets will be connected by an edge if their intersection is the empty set (see Figure 2). Let (S_{ij}, S_{kl}) be any pair of V_p . Then either $S_{ij} \cap S_{kl} = \emptyset$, or $S_{ij} \cap S_{kl}$ is a singleton set. In the former case, $i, j \neq k, l$ and $d(S_{ij}, S_{kl}) = 1$, and in the later case, $i = k$, or $i = l$, or $j = k$, or $j = l$ and $d(S_{ij}, S_{kl}) = 2$. Now, we discuss two cases.

Case 1. When $S_{ij} \cap S_{kl} = \emptyset$. Then for each element $A \in V_1(S_{ij})$, $A \cap S_{kl} \neq \emptyset$, and for each element $B \in V_1(S_{kl})$, $B \cap S_{ij} \neq \emptyset$. It follows that each A and each B resolves S_{ij} and S_{kl} . Further, for each $S_{ab} \in V(G) - (V_1(S_{ij}) \cup V_1(S_{kl}))$, $S_{ab} \cap S_{ij} \neq \emptyset$ as well as $S_{ab} \cap S_{kl} \neq \emptyset$, and hence $S_{ij}, S_{kl} \in V_2(S_{ab})$. Thus, the resolving neighborhood of the pair (S_{ij}, S_{kl}) is $V_1(S_{ij}) \cup V_1(S_{kl})$ having six elements. Thus, for $X \in V(G)$, by Lemma 2.3, we have

$$r_X(S_{ij}, S_{kl}) = \begin{cases} \frac{1}{6} & \text{if } X \in V_1(S_{ij}) \cup V_1(S_{kl}), \\ 0 & \text{otherwise.} \end{cases}$$

Case 2. When $S_{ij} \cap S_{kl} \neq \emptyset$. Then (i) $V_1(S_{ij}) \cap V_1(S_{kl}) = \{S_{ab}\}$, where $a, b \neq i, j, k, l$, and hence S_{ij}, S_{kl} belong to the same partite set $V_1(S_{ab})$, (ii) $|V_2(S_{ij}) \cap V_2(S_{kl})| = 3$ and for each $A \in V_2(S_{ij}) \cap V_2(S_{kl})$, S_{ij}, S_{kl} belong to the same partite set $V_2(A)$. Thus $R(S_{ij}, S_{kl}) = V(G) - [(V_2(S_{ij}) \cap V_2(S_{kl})) \cup \{S_{ab}\}]$. Hence, for $X \in V(G)$, by Lemma 2.3, we have

$$r_X(S_{ij}, S_{kl}) = \begin{cases} \frac{1}{6} & \text{if } X \notin V(G) - [(V_2(S_{ij}) \cap V_2(S_{kl})) \cup \{S_{ab}\}], \\ 0 & \text{otherwise.} \end{cases}$$

Note that, for any 2-element subset $X \in V(G)$, $r_X(Y, Z) = \frac{1}{6}$ for all $(Y, Z) \in R(X)$. Thus, by Proposition 3.3, $ar_X(G) = \frac{1}{6}$ for all $X \in V(G)$. It completes the proof. \square

Theorem 3.7. *Let G be a path on $n \geq 3$ vertices. Then*

$$\mathcal{R}(G) = \begin{cases} \sum_{i=1}^{\frac{n}{2}} \frac{2n^2-3n-4i+4}{n(n-1)^2-2(n-1)(i-1)} & \text{if } n \text{ is even,} \\ \frac{2n^2-3n-1}{2n(n-1)^2} + \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \frac{2n^3-3n^2-4n(i-1)+1}{n^2(n-1)^2-2n(n-1)(i-1)} & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Let $G : v_1, v_2, \dots, v_n$ be a path on $n \geq 3$ vertices and let $(v_i, v_j) \in V_p$. Then

$$R(v_i, v_j) = \begin{cases} V(G) & \text{if } i+j \text{ is odd,} \\ V(G) - \{v_{\frac{i+j}{2}}\} & \text{if } i+j \text{ is even.} \end{cases}$$

Hence for $w \in V(G)$,

$$r_w(v_i, v_j) = \begin{cases} 0 & \text{for even } i+j, \text{ and } w = v_{\frac{i+j}{2}}, \\ \frac{1}{n-1} & \text{for even } i+j, \text{ and for all } w \in V(G) - \{v_{\frac{i+j}{2}}\}, \\ \frac{1}{n} & \text{for odd } i+j, \text{ and for all } w \in V(G). \end{cases}$$

Now, since

$$\Pi_v = \begin{cases} \{V_j(v) ; 1 \leq j \leq n-i\} & \text{for } v \in \{v_i, v_{n-i+1}\}, 1 \leq i \leq \lfloor \frac{n}{2} \rfloor, \\ \{V_j(v) ; 1 \leq j \leq i-1\} & \text{for } v = v_i \text{ and } i = \lceil \frac{n}{2} \rceil \text{ for odd } n. \end{cases}$$

So Π_{v_i} and $\Pi_{v_{n-i+1}}$ contain $(i-1)$ 2-element subsets of $V(G)$ for each $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$ and $i = \lceil \frac{n}{2} \rceil$. It follows that each v_i and v_{n-i+1} do not resolve $i-1$ pairs of V_p , and hence

$$|R(v_i)| = |R(v_{n-i+1})| = \binom{n}{2} - (i-1).$$

The resolvent neighborhood of each vertex of G consists of two types of pairs of V_p : the pairs (v_a, v_b) for which $a+b$ is even, we refer them the pairs of type-I; the pairs (v_a, v_b) for which $a+b$ is odd, we refer them the pairs of type-II. In above, we have proved that the resolving share of the vertices of G for the pairs of type-I is $\frac{1}{n-1}$ and for the pairs of type-II is $\frac{1}{n}$. We discuss the following two cases:

Case 1. (n is even) Since the sum $a+b$ of two natural numbers a, b will be even when both a and b are even, or both a and b are odd, and there are $\frac{n}{2}$ even and $\frac{n}{2}$ odd natural numbers in the set $\{1, 2, \dots, n\}$ in this case. So V_p contains $2\binom{\frac{n}{2}}{2} = \frac{1}{4}n(n-2)$ pairs of type-I and $\binom{n}{2} - \frac{1}{4}n(n-2) = \frac{1}{4}n^2$ pairs of type-II. Out of $\frac{1}{4}n(n-2)$ pairs of type-I, those $\frac{1}{4}n(n-2) - (i-1)$ pairs belong to $R(v_i)$ and $R(v_{n-i+1})$ for which the sum of the indices of the vertices in the pairs is not equal to $2i$ or $2(n-i+1)$, respectively. Also, all the $\frac{1}{4}n^2$ pairs of type-II belong to $R(v_i)$ and $R(v_{n-i+1})$, where $1 \leq i \leq \frac{n}{2}$. It follows that for each $1 \leq i \leq \frac{n}{2}$,

$$ar_{v_i}(G) = \frac{2}{n(n-1) - 2(i-1)} \left(\frac{n(n-2)}{4(n-1)} - \frac{i-1}{n-1} + \frac{n}{4} \right) = ar_{v_{n-i+1}}(G).$$

Case 2. (n is odd) Since there are $\frac{n-1}{2}$ even and $\frac{n+1}{2}$ odd natural numbers in the set $\{1, 2, \dots, n\}$ in this case. So V_p contains $\binom{\frac{n-1}{2}}{2} + \binom{\frac{n+1}{2}}{2} = \frac{1}{4}(n-1)^2$ pairs of type-I and $\binom{n}{2} - \frac{1}{4}(n-1)^2 = \frac{1}{4}(n^2-1)$ pairs of type-II. Out of $\frac{1}{4}(n-1)^2$ pairs of type-I, those $\frac{1}{4}(n-1)^2 - (i-1)$ pairs belong to $R(v_i)$ and $R(v_{n-i+1})$ for which the sum of the indices of the vertices in the pairs is not equal to $2i$ or $2(n-i+1)$, respectively. Also, all the $\frac{1}{4}(n^2-1)$ pairs of type-II belong to $R(v_i)$ and $R(v_{n-i+1})$, where $1 \leq i \leq \lceil \frac{n}{2} \rceil$. It follows that for each $1 \leq i \leq \lceil \frac{n}{2} \rceil$,

$$ar_{v_i}(G) = \frac{2}{n(n-1) - 2(i-1)} \left(\frac{n-1}{4} - \frac{i-1}{n-1} + \frac{n^2-1}{4n} \right) = ar_{v_{n-i+1}}(G).$$

By adding the average resolving shares of all the vertices of G in the both above cases, one can easily derive the stated resolving topological index. \square

An even path in a graph G is such whose length is even, and a path is odd if its length is odd.

Theorem 3.8. *Let G be a cycle on $n \geq 3$ vertices. Then*

$$\mathcal{R}(G) = \begin{cases} \frac{n(n-1)}{n^2-2n+2} & \text{when } n \text{ is even,} \\ \frac{n}{n-1} & \text{when } n \text{ is odd.} \end{cases}$$

Proof. We consider the following two cases:

Case 1. (n in even) Let $G : v_1, v_2, \dots, v_n, v_1$ be a cycle on even $n \geq 4$ vertices and let D be the diameter of G . Then for $(v_i, v_j) \in V_p$ ($i \neq j$),

$$R(v_i, v_j) = \begin{cases} V(G) & \text{when } i+j \text{ is odd,} \\ V(G) - \{v_{\frac{i+j}{2}}, v_{\frac{i+j}{2}+D}\} & \text{when } i+j \text{ is even.} \end{cases}$$

Hence for $w \in V(G)$,

$$r_w(v_i, v_j) = \begin{cases} 0 & \text{for even } i+j, \text{ and } w \in \{v_{\frac{i+j}{2}}, v_{\frac{i+j}{2}+D}\}, \\ \frac{1}{n-2} & \text{for even } i+j, \text{ and for all } w \in V(G) - \{v_{\frac{i+j}{2}}, v_{\frac{i+j}{2}+D}\}, \\ \frac{1}{n} & \text{for odd } i+j, \text{ and for all } w \in V(G). \end{cases}$$

Since for each $1 \leq i \leq n$, the distance partition $\Pi_{v_i} = \{V_j(v_i) ; 1 \leq j \leq D\}$ contains $(D-1)$ 2-element subsets of $V(G)$, so each v_i do not resolve $D-1$ pairs of V_p , and hence

$$|R(v_i)| = \binom{n}{2} - (D-1) = \frac{n(n-1) - 2(D-1)}{2}.$$

Note that, the resolvent neighborhood of each vertex of G consists of two types of pairs of V_p : the pairs (v_a, v_b) for which $a+b$ is even, we refer them the pairs of type-I; the pairs (v_a, v_b) for which $a+b$ is odd, we refer them the pairs of type-II. In above, we have proved that the resolving share of the vertices of G for the pairs of type-I is $\frac{1}{n-2}$ and for the pairs of type-II is $\frac{1}{n}$. Since the sum $a+b$ of two natural

numbers a, b will be even when both a and b are even, or both a and b are odd, and there are $\frac{n}{2}$ even and $\frac{n}{2}$ odd natural numbers in the set $\{1, 2, \dots, n\}$ in this case. So V_p contains $2\binom{\frac{n}{2}}{2} = \frac{1}{4}n(n-2)$ pairs of type-I and $\binom{n}{2} - \frac{1}{4}n(n-2) = \frac{1}{4}n^2$ pairs of type-II. Out of $\frac{1}{4}n(n-2)$ pairs (v_a, v_b) of type-I, those pairs belong to $R(v_i)$ for which $i \neq \frac{a+b}{2}$ and $i \neq \frac{a+b}{2} + D$, and there are $\frac{1}{4}n(n-2) - (D-1)$ such pairs, where $1 \leq i \leq \frac{n}{2}$. Also, all the $\frac{1}{4}n^2$ pairs of type-II belong to $R(v_i)$, where $1 \leq i \leq n$. It follows that for each $1 \leq i \leq n$,

$$ar_{v_i}(G) = \frac{2}{n(n-1) - 2(D-1)} \left(\frac{n(n-2)}{4(n-2)} - \frac{D-1}{n-2} + \frac{n^2}{4n} \right) = \frac{n-1}{n^2 - 2n + 2},$$

because $D = \frac{n}{2}$. It completes the proof.

Case 2. (n is odd) Let (u, v) be any pair of V_p and let l be the length of the even $u - v$ path in G . Then there exists a vertex x in G such that $u, v \in V_{\frac{l}{2}}(x) \in \Pi_x$. It follows, by Lemma 2.3, that

$$r_w(u, v) = \begin{cases} 0 & \text{for } w = x, \\ \frac{1}{n-1} & \text{for all } w \in V(G) - \{x\}. \end{cases}$$

Since the resolving share of each vertex w of G for every pair in V_p (and hence for every pair in the resolvent neighborhood of w) is $\frac{1}{n-1}$. So Proposition 3.3 yields that $ar_w(G) = \frac{1}{n-1}$ for all $w \in V(G)$, and it concludes the proof. \square

Theorem 3.9. *The resolving topological index of a complete graph on at least two vertices is $\frac{|G|}{2}$.*

Proof. Since $\text{diam}(G) = 1$, so Theorem 2.14 implies that for every pair $(u, v) \in V_p$ and for each $w \in V(G)$,

$$r_w(u, v) = \begin{cases} \frac{1}{2} & \text{if } w \in \{u, v\}, \\ 0 & \text{if } w \notin \{u, v\}. \end{cases}$$

Hence $\mathcal{R}(G) = \frac{|G|}{2}$ because $ar_w(G) = \frac{1}{2}$ for all $w \in V(G)$. \square

Theorem 3.10. *Let G be a complete k -partite graph K_{n_1, n_2, \dots, n_k} , where each $n_i \geq 2$, $1 \leq i \leq k$ and $k \geq 2$. Then*

$$\mathcal{R}(G) = \sum_{i=1}^k n_i \left(\frac{n_i - 1}{n_i} + \sum_{\substack{t=1 \\ t \neq i}}^k n_t \right)^{-1} \left(\frac{n_i - 1}{2n_i} + \sum_{\substack{t=1 \\ t \neq i}}^k \frac{n_t}{n_i + n_t} \right).$$

Proof. Let the partite sets of G are $V_i = \{v_1^i, v_2^i, \dots, v_{n_i}^i\}$, where $n_i \geq 2$ and $1 \leq i \leq k$. Let $(x, y) \in V_p$. If $x, y \in V_i$ for any $1 \leq i \leq k$, then $\Pi_x - \{y\} = \Pi_y - \{x\}$. It follows, by Lemmas 2.4 and 2.5, that

$$r_w(x, y) = \begin{cases} \frac{1}{2} & \text{if } w \in \{x, y\}, \\ 0 & \text{if } w \notin \{x, y\}. \end{cases}$$

If $x \in V_i$ and $y \in V_{j \neq i}$, where $1 \leq i, j \leq k$. Then for each $u \in V_i - \{x\}, v \in V_j - \{y\}$ and $z \in V_l$ (for all $1 \leq l \leq k$ and $l \neq i, j$), $x \in V_1(v) \cup V_2(u), y \in V_1(u) \cup V_2(v)$ and $x, y \in V_1(z)$. It follows that the resolving neighborhood of (x, y) is $V_i \cup V_j$. Thus, by Lemma 2.3,

$$r_w(x, y) = \begin{cases} \frac{1}{n_i + n_j} & \text{if } w \in V_i \cup V_j, \\ 0 & \text{otherwise.} \end{cases}$$

For each $1 \leq i \leq k$ and for any $v_j^i \in V_i$ ($1 \leq j \leq n_i$),

$$R(v_j^i) = \{(v_j^i, v_l^i), (v_a^i, v_b^t); 1 \leq l \neq j \leq n_i, 1 \leq a \leq n_i, 1 \leq b \leq n_t, 1 \leq t \neq i \leq k\}.$$

Since

$$r_{v_j^i}(v_j^i, v_l^i) = \frac{1}{2} \quad \text{for all } 1 \leq l \neq j \leq n_i, \text{ and}$$

$$r_{v_j^i}(v_a^i, v_b^t) = \frac{1}{n_i + n_t} \quad \text{for all } 1 \leq a \leq n_i, 1 \leq b \leq n_t, 1 \leq t \neq i \leq k.$$

So for each $1 \leq i \leq k$ and for each $1 \leq j \leq n_i$,

$$\begin{aligned} ar_{v_j^i}(G) &= \frac{1}{|R(v_j^i)|} \left(\sum_{\substack{l=1 \\ l \neq j}}^{n_i} \frac{1}{2} + \sum_{\substack{t=1 \\ t \neq i}}^k \sum_{a=1}^{n_i} \sum_{b=1}^{n_t} \frac{1}{n_i + n_t} \right) \\ &= \left[(n_i - 1) + n_i \sum_{\substack{t=1 \\ t \neq i}}^k n_t \right]^{-1} \left[\frac{n_i - 1}{2} + n_i \sum_{\substack{t=1 \\ t \neq i}}^k \frac{n_t}{n_i + n_t} \right] \\ &= \left[n_i \left(1 + \sum_{\substack{t=1 \\ t \neq i}}^k n_t \right) - 1 \right]^{-1} \left[n_i \left(\frac{1}{2} + \sum_{\substack{t=1 \\ t \neq i}}^k \frac{n_t}{n_i + n_t} \right) - \frac{1}{2} \right]. \end{aligned}$$

Now, by taking the summation of the average resolving shares of the vertices of all the partite sets of $V(G)$, we get the required result. \square

Theorem 3.11. *For each wheel graph W_n ($n \geq 6$), the resolving topological index is $\frac{(n-3)(n^2+8)}{2(n-2)(4n-13)}$.*

Proof. A wheel graph W_n is the join of a cycle C_{n-1} and the vertex c (called the central vertex of the wheel). First note that, $\Pi_c = \{V_1(c) = V(C_{n-1})\}$ and for each $v \in V(C_{n-1})$, $\Pi_v = \{V_1(v), V_2(v)\}$ with $|V_1(v)| = 3$ and $|V_2(v)| = n - 4$ since the diameter of wheel is two. We consider two cases for any $(x, y) \in V_p$.

Case 1. When $d(x, y) = 1$. If $y = c$ and $x \in V(C_{n-1})$, then each vertex u of C_{n-1} such that $d(u, x) = 2$ belongs to the resolving neighborhood of (x, y) . Also,

for each $v \in V_1(x) - \{c\}$, x and y belong to the same partite set of Π_v . Thus $R(x, y) = V_2(x) \cup \{x, y\}$, and hence Lemma 2.3 yields that

$$r_w(x, y) = \begin{cases} 0 & \text{if } w \in V_1(x) - \{c\}, \\ \frac{1}{n-2} & \text{otherwise.} \end{cases}$$

If both x and y belong to $V(C_{n-1})$, then $x \in V_1(y)$ and $y \in V_1(x)$. In this case, $x, y \in V_1(c)$ and for each $z \in V_2(x) \cap V_2(y)$, $x, y \in V_2(z)$. Also, for $u \in V_1(x) - \{y, c\}$ and for $v \in V_1(y) - \{x, c\}$, $x \in V_2(v)$ and $y \in V_2(u)$. It follows that $R(x, y) = \{u, v, x, y\}$, and hence, by Lemma 2.3, we have

$$r_w(x, y) = \begin{cases} 0 & \text{if } w \in \{c\} \cup (V_2(x) \cap V_2(y)), \\ \frac{1}{4} & \text{otherwise.} \end{cases}$$

Case 2. When $d(x, y) = 2$, then $x, y \in V(C_{n-1})$ and $x \in V_2(y), y \in V_2(x)$. There are two subcases to discuss.

Subcase 2.1. When $|V_1(x) \cap V_1(y)| = 1$. In this case, for each $u \in V_1(x) - \{c\}$ and for each $v \in V_1(y) - \{c\}$, $x \in V_2(v)$ and $y \in V_2(u)$. So $R(x, y) = V_2(x) \nabla V_2(y)$ and $|R(x, y)| = 6$ (the symbol $X \nabla Y$ denotes the symmetric difference of two sets X and Y). Further, for each $z \notin V_2(x) \nabla V_2(y)$, x, y belong to the same partite set of Π_z . It follows, by Lemma 2.3, that

$$r_w(x, y) = \begin{cases} 0 & \text{if } w \notin V_2(x) \nabla V_2(y), \\ \frac{1}{6} & \text{otherwise.} \end{cases}$$

Subcase 2.2. When $|V_1(x) \cap V_1(y)| = 2$. In this case, for $u \in V_1(x) - (V_1(x) \cap V_1(y))$ and for $v \in V_1(y) - (V_1(x) \cap V_1(y))$, $x \in V_2(v)$ and $y \in V_2(u)$. So $R(x, y) = V_2(x) \nabla V_2(y) = \{u, v, x, y\}$. Moreover, for each $z \notin \{u, v, x, y\}$, x, y belong to the same partite set of Π_z . It concludes, by Lemma 2.3, that

$$r_w(x, y) = \begin{cases} 0 & \text{if } w \notin \{u, v, x, y\}, \\ \frac{1}{4} & \text{otherwise.} \end{cases}$$

Since $R(c) = \{(c, v) ; v \in V(C_{n-1})\}$ and each $v \in V(C_{n-1})$ does not resolve the pairs (a, b) and (x, y) for all distinct $a, b \in V_1(v)$ and for all distinct $x, y \in V_2(v)$. It follows that $|R(c)| = n - 1$ and for each $v \in V(C_{n-1})$, $|R(v)| = \binom{n}{2} - \binom{3}{2} - \binom{n-4}{2} = 4n - 13$. For $v \in V(C_{n-1})$, let $V_1(v) = \{c, u, w\}$. Then out of $4n - 13$ pairs in $R(v)$, (i) $n - 3$ pairs are of the form (c, a) , $a \in V(W_n) - V_1(v)$, and the resolving share of v for all these pairs is $\frac{1}{n-2}$; (ii) $n - 2$ pairs are of the form (v, b) , $b \in V(W_n) - \{c, v\}$, and the resolving share of v for 4 out of these $n - 2$ pairs is $\frac{1}{4}$ and is $\frac{1}{6}$ for the remaining $n - 6$ pairs; (iii) $2(n - 4)$ pairs are of the form (u, d) and (w, d) , $d \in V(W_n) - \{c, u, v, w\}$, and the resolving share of v for 4 out of these $2(n - 4)$ pairs is $\frac{1}{4}$ and is $\frac{1}{6}$ for the remaining $2(n - 6)$ pairs. Also, the resolving share of the central vertex c for each pair in $R(c)$ is $\frac{1}{n-2}$. Hence, by Proposition 3.3, $ar_c(W_n) = \frac{1}{n-2}$ and for each

$$v \in V(C_{n-1}),$$

$$\begin{aligned} ar_v(W_n) &= \frac{1}{4n-13} \left((n-3)\frac{1}{n-2} + 4\left(\frac{1}{4}\right) + (n-6)\frac{1}{6} + 4\left(\frac{1}{4}\right) + 2(n-6)\frac{1}{6} \right) \\ &= \frac{n^2 - 2n - 2}{2(n-2)(4n-13)}. \end{aligned}$$

Hence

$$\mathcal{R}(W_n) = \frac{1}{n-2} + (n-1) \frac{n^2 - 2n - 2}{2(n-2)(4n-13)} = \frac{(n-3)(n^2+8)}{6(n-2)(4n-13)}.$$

□

Theorem 3.12. *The resolving topological index of a friendship graph F_n is $\frac{2n^3-n^2+4n-4}{4n(3n-2)}$, where $n \geq 2$.*

Proof. A friendship graph F_n is the join $K_1 + G$ and having $2n + 1$ vertices, where K_1 is a graph having only one vertex c (called the central vertex) and G is the graph obtain by taking the union of n copies of the path P_2 . For each $u \in V(G)$, $\Pi_u = \{V_1(u), V_2(u)\}$ with $|V_1(u)| = 2$ and $|V_2(u)| = 2(n-1)$ since the diameter of F_n is two. Let $v \in V(F_n) - \{u\}$. If $v \in V_1(u)$ and $v = c$, then for the vertex $x \in V_1(u) - \{v\}$, u and v belong to the same partite set of Π_x . Also, for each $y \in V_2(u)$, $v \in V_1(y)$ and $u \in V_2(y)$. Hence, together with Lemma 2.3 and above discussion, we have

$$r_w(u, v) = \begin{cases} 0 & \text{if } w = x, \\ \frac{1}{2n} & \text{otherwise.} \end{cases}$$

If $v \in V_1(u)$ and $v \neq c$, then u and v are the vertices of the same copy of P_2 and $\Pi_u - \{v\} = \Pi_v - \{u\}$. It follows, by Lemmas 2.4 and 2.5, that

$$r_w(u, v) = \begin{cases} 0 & \text{if } w \notin \{u, v\}, \\ \frac{1}{2} & \text{if } w \in \{u, v\}. \end{cases}$$

If $v \in V_2(u)$, then for $a \in V_1(u) - \{c\}$ and for $b \in V_1(v) - \{c\}$, $u \in V_2(b)$ and $v \in V_2(a)$. Also, for all $x \notin (V_1(u) \cup V_1(v)) - \{c\}$, u and v belong to the same partite set of Π_x . Thus $R(u, v) = \{a, b, u, v\}$, and hence

$$r_w(u, v) = \begin{cases} 0 & \text{if } w \in \{c\} \cup (V_1(u) \cup V_1(v)), \\ \frac{1}{4} & \text{otherwise.} \end{cases}$$

For any $u \in V(G)$, let $V_1(u) = \{a, c\}$. Then

$$\begin{aligned} R(u) &= V_p - (\{(a, c)\} \cup \{(x, v) ; x, v \in V_2(u)\}) \\ &= \{(u, w), (a, y), (c, y) ; w \in V(F_n) - \{u\}, y \in V(F_n) - \{a, c, u\}\}. \end{aligned}$$

In $R(u)$, (i) the number of pairs of the form (u, w) is $2n$, and the resolving share of u for the pair (u, a) is $\frac{1}{2}$, for the pair (u, c) is $\frac{1}{2n}$ and for the remanding $2(n-1)$ pairs is $\frac{1}{4}$; (ii) the number of pairs of the form (a, y) is $2(n-1)$, and the resolving share

of u for all these pairs is $\frac{1}{4}$; (iii) the number of pairs of the form (c, y) is $2(n-1)$, and the resolving share of u for all these pairs is $\frac{1}{2n}$. Thus $|R(u)| = 2(3n-2)$, and hence

$$\begin{aligned} ar_u(F_n) &= \frac{1}{2(3n-2)} \left(\frac{1}{2} + \frac{1}{2n} + 2(n-1)\frac{1}{4} + 2(n-1)\frac{1}{4} + 2(n-1)\frac{1}{2n} \right) \\ &= \frac{(n+1)(2n-1)}{4n(3n-2)}. \end{aligned}$$

Since $R(c) = \{(c, v) ; v \in V(G)\}$ and the resolving share of c for all these $2n$ pairs is $\frac{1}{2n}$. So, Proposition 3.3 concludes that $ar_c(F_n) = \frac{1}{2n}$. It concludes that

$$\mathcal{R}(F_n) = \frac{1}{2n} + (n-1) \frac{(n+1)(2n-1)}{4n(3n-2)} = \frac{2n^3 - n^2 + 4n - 4}{4n(3n-2)}.$$

□

4. CONCLUDING REMARKS

We investigated the amount of the resolving done by a vertex v of a graph G for every pair of vertices of G , called the resolving share of v , and then we established some related results. We also quantified the average of the amount of resolving done by v in G , and we called it the average resolving share of v . Using average resolving share of each vertex of G , we associated a distance-based topological index with the graph G , which describes the topology of that graph with respect to the total resolving done by each vertex of that graph, and we called this topological index, the resolving topological index. Then, by computing the resolving shares and average resolving shares of all the vertices, we worked out the resolving topological indices of certain well-known graphs such as the Petersen graph, paths, cycles, complete graphs, complete k -partite graphs, wheel graphs and friendship graph. The work done in this paper is a revelation for the researchers working with resolvability to determine, in different graphical structures, how they have the topology according to the resolving done by their vertices.

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